Bayesian updating of statistical parameters and probability models for ice peak loads

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Objective

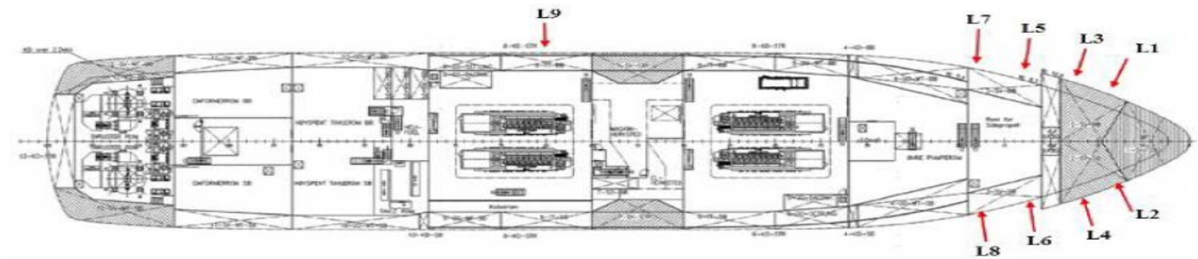
- Present probabilistic models for peak ice loads acting on a ship in arctic areas

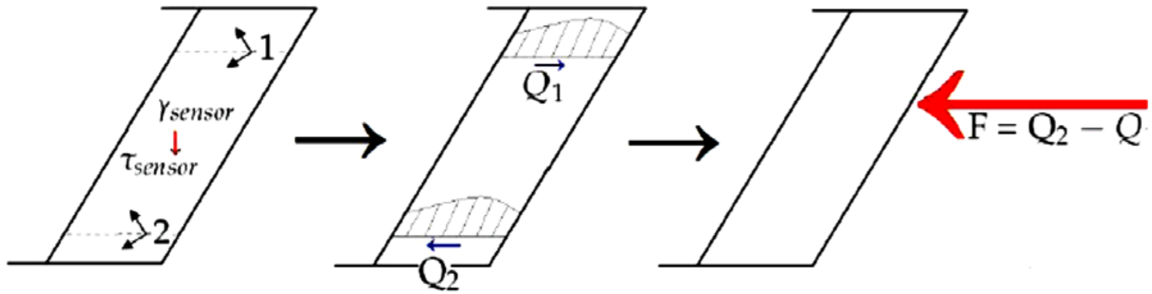
- Illustrate Bayesian updating of statistical parameters for a simplistic data set

- Bayesian updating for mixture of different models which are members of a selected “model universe”

- Compute probability of failure for a fixed capacity threshold for different probability models (based on posterior distribution functions)

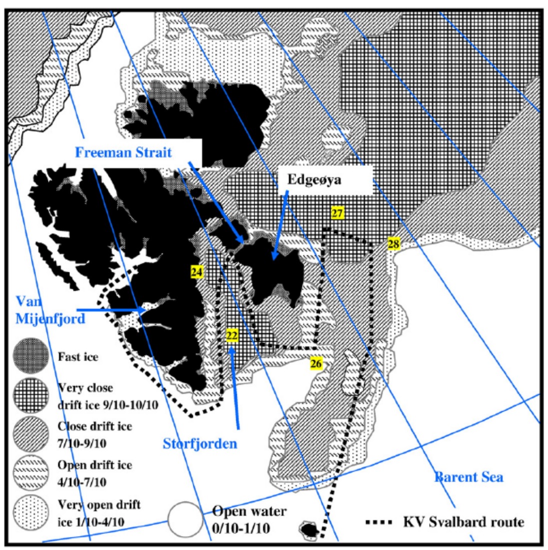
Ice-induced forces: Data from KV Svalbard 2007 expedition.





(a) (b) (c)

- The shear strain measured is converted into shear stress.
- The total shear force Q on the frame obtained by integration.
- The ice force F computed from the difference between the shear forces at the upper and lower part of the frame Q2-Q1.



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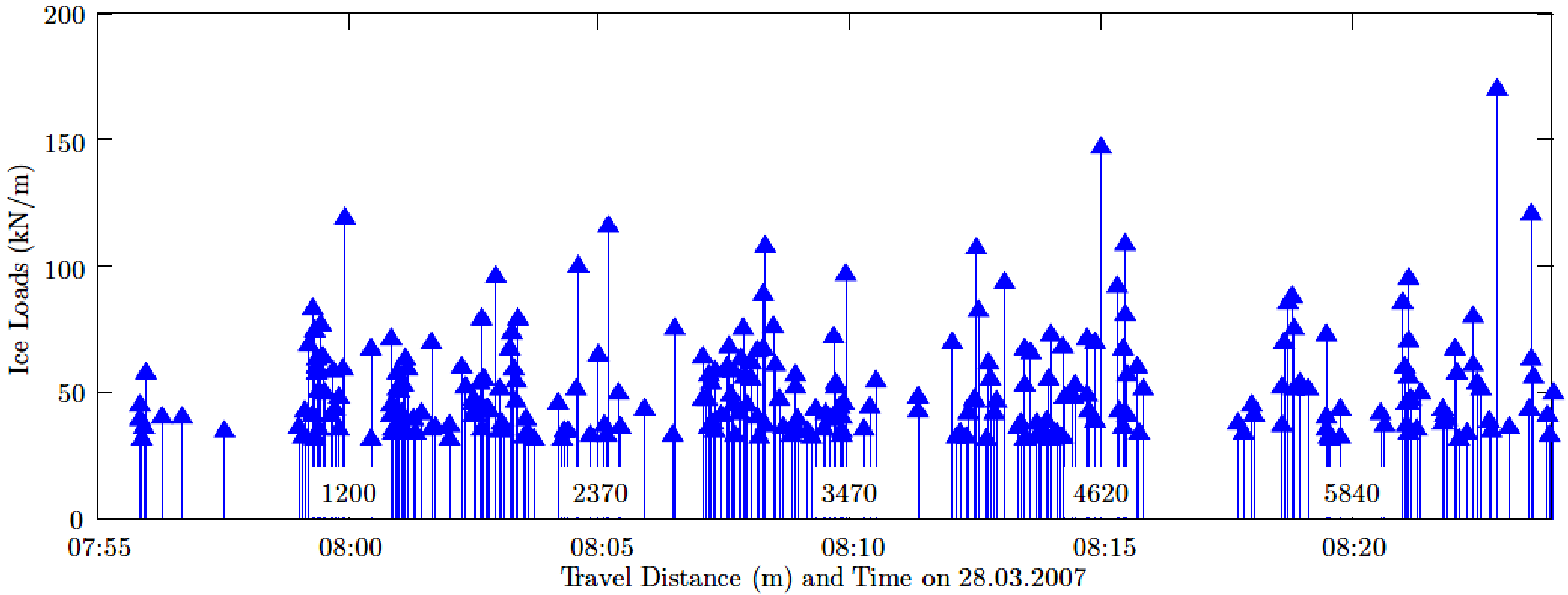
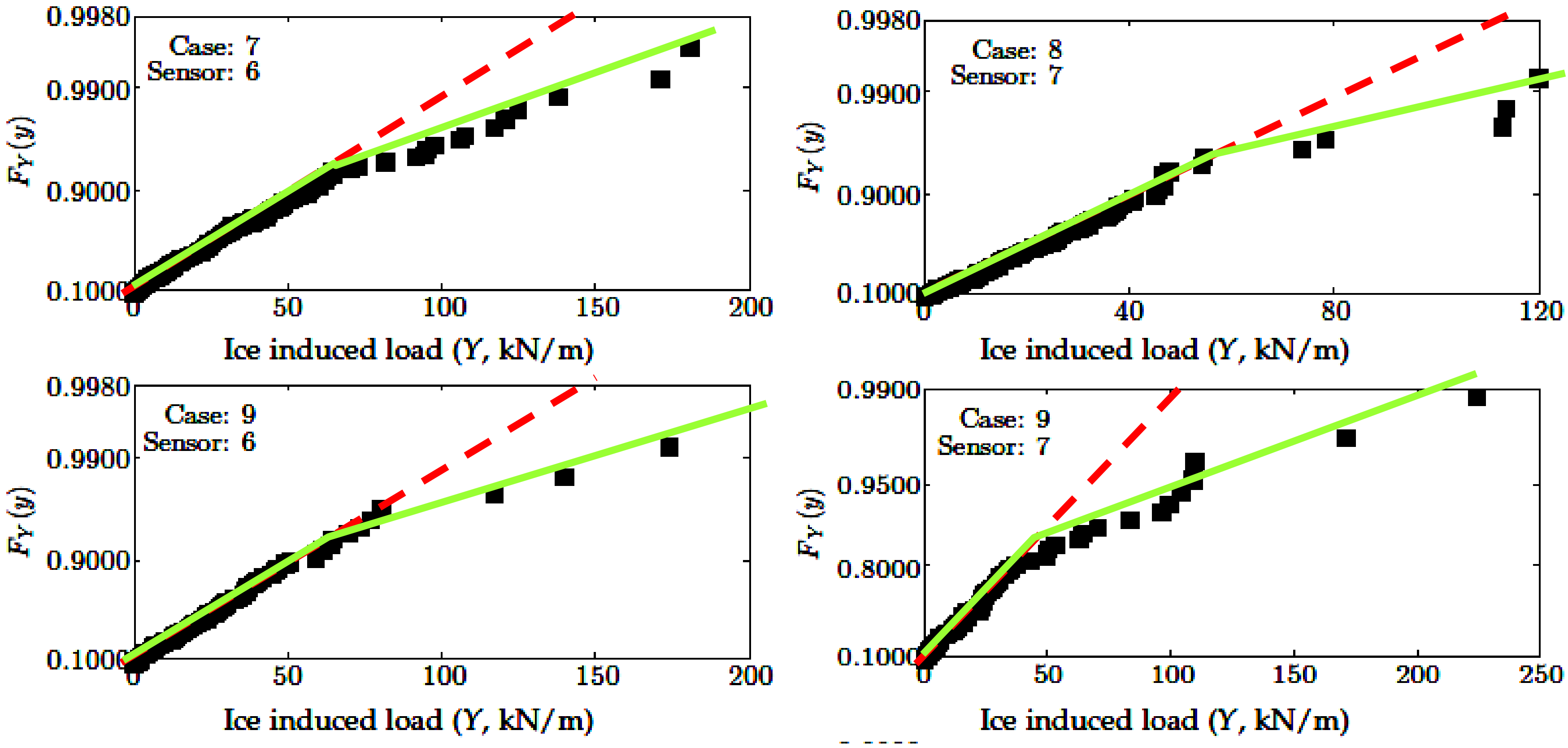
Initial/parent distribution of the ice load peak process.



What is the statistical model for this process?

A single or a compound population?



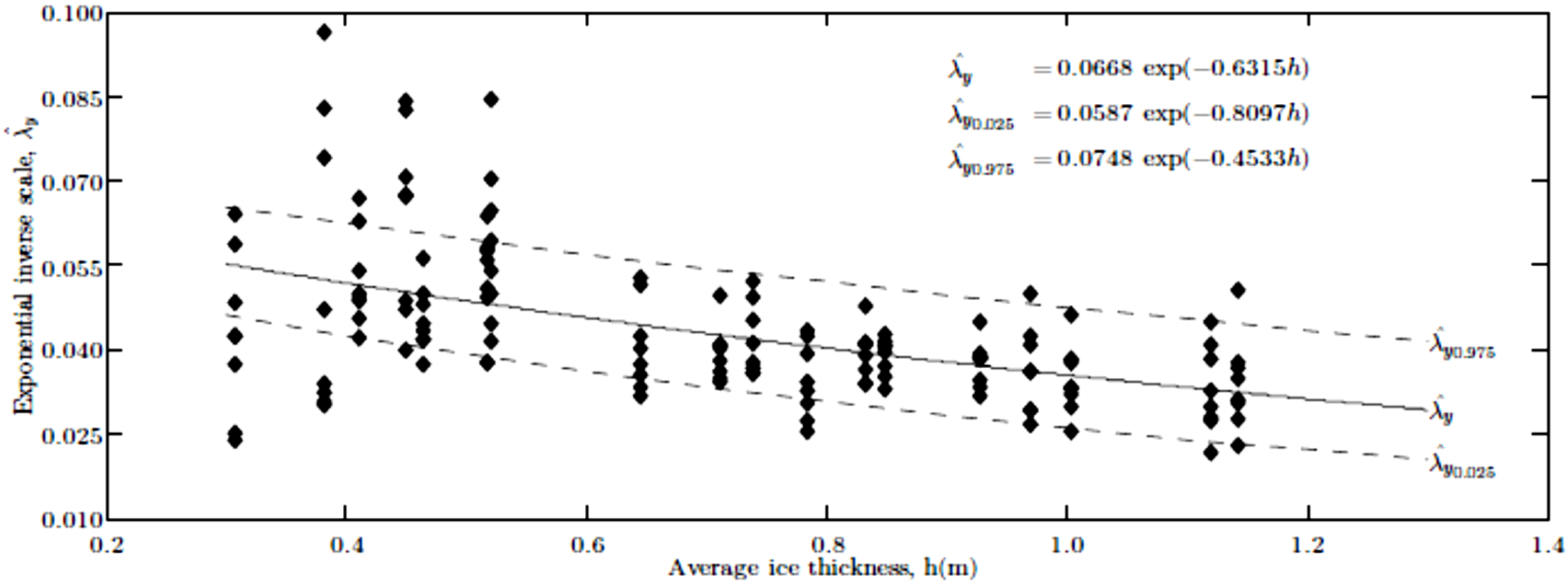
A ***generalized model is proposed***: a proportional combination of two one-parameter exponential models.

INITIAL

Conditional distribution for a given stationary condition

EXP

$$F_{Y|H_i}(y|h) = 1 - \exp(-\lambda_y(h)\, y)$$



Inverse scale down:

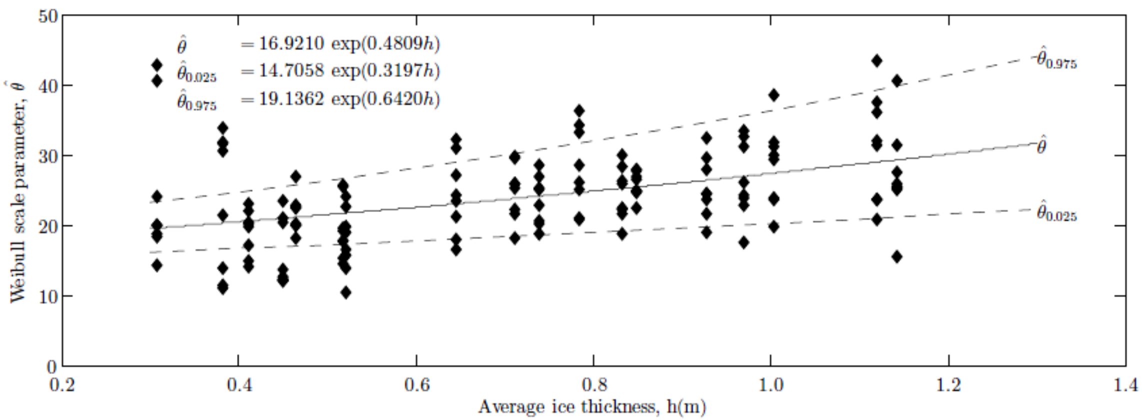
thicker tail

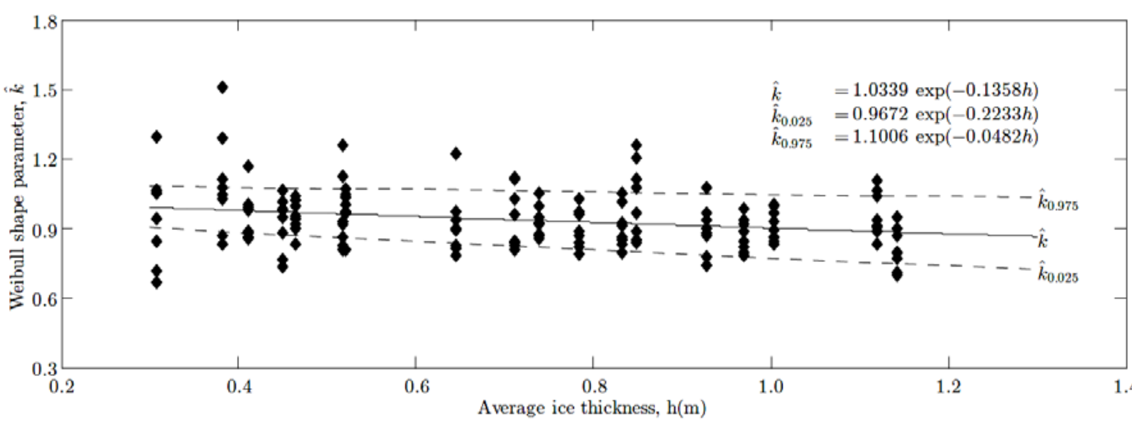
(***more severe*** ice load)

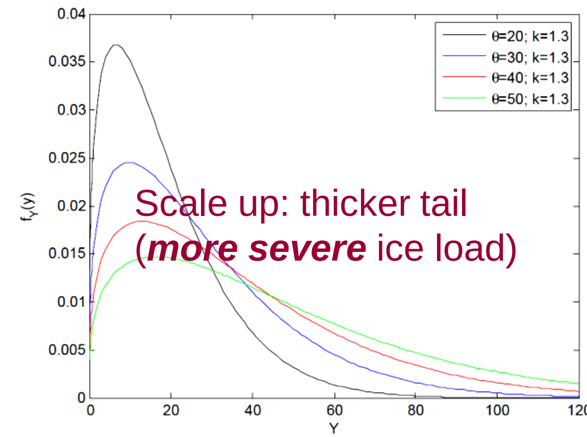
Conditional distribution for a given stationary condition

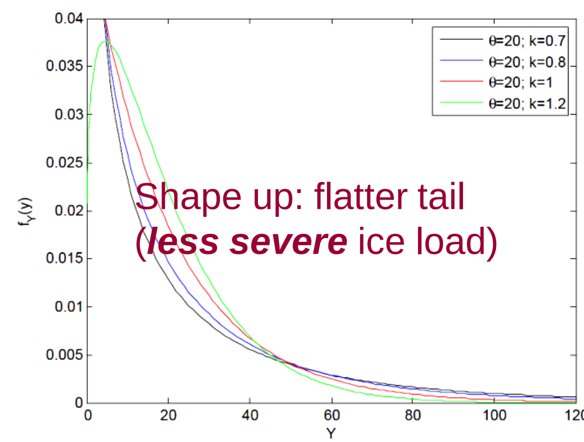
Weibull

$$F_{Y|H_i}(y|h) = 1 - \exp\left\{-\left(\frac{y}{\theta(h)}\right)^{k(h)}\right\}$$









Conditional distribution for a given stationary condition

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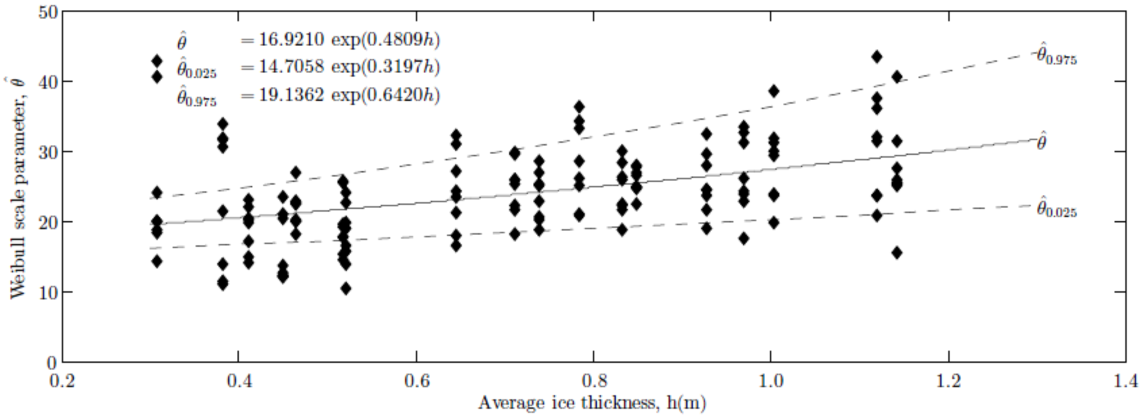
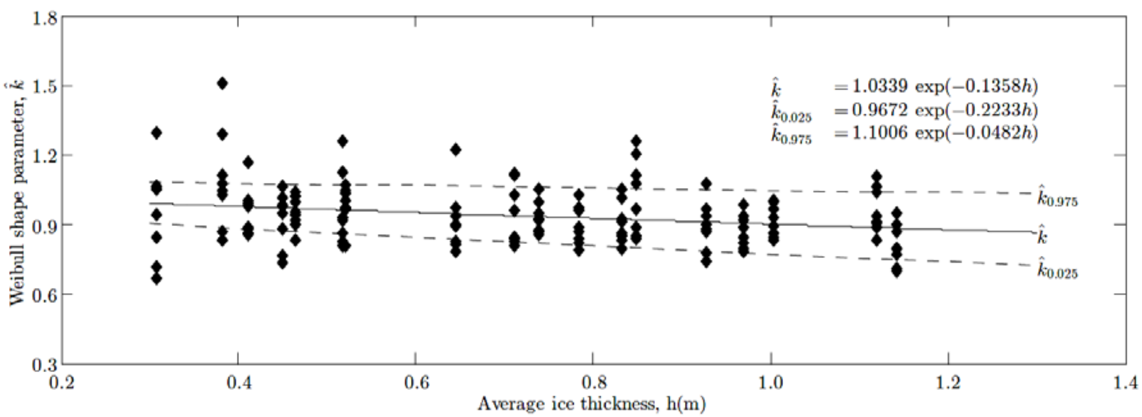
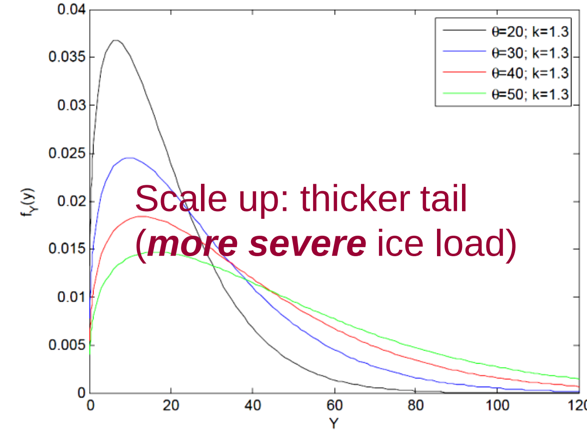
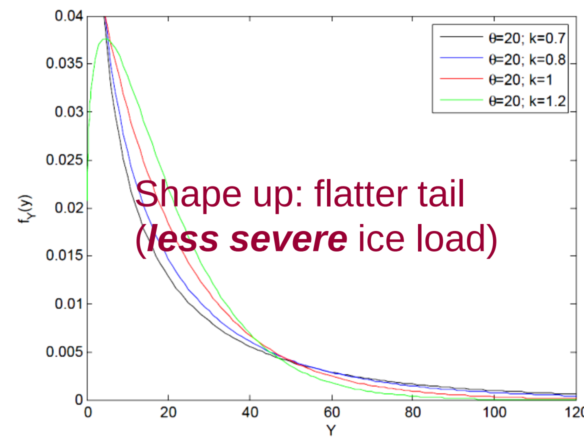






Scale up: thicker tail
(***more severe*** ice load)

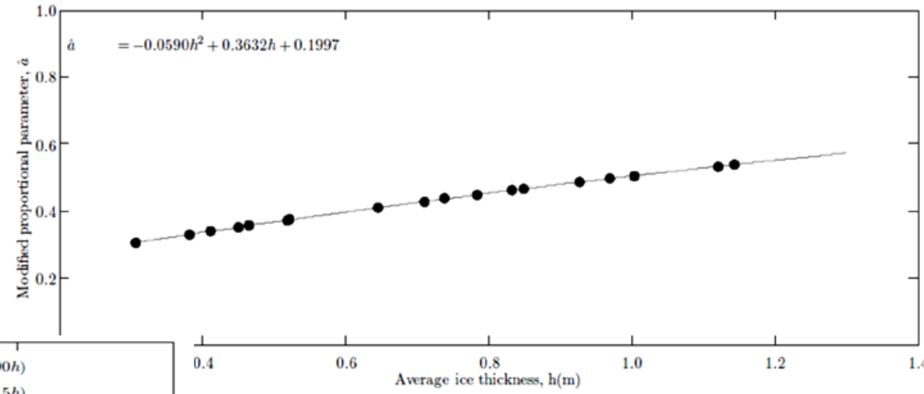


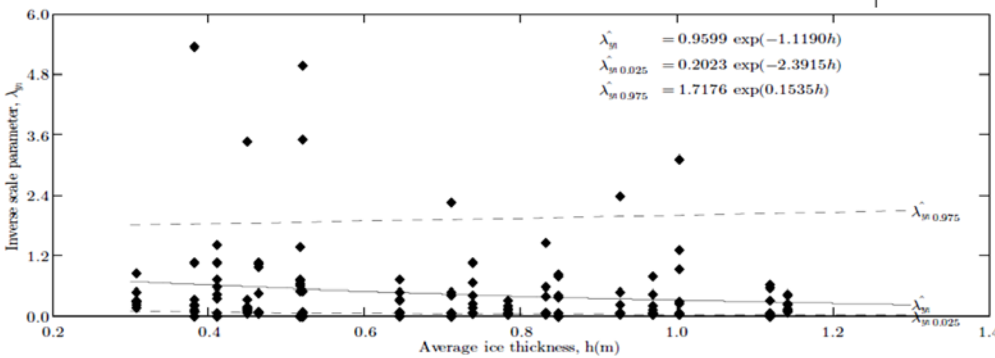
Shape up: flatter tail
(***less severe*** ice load)

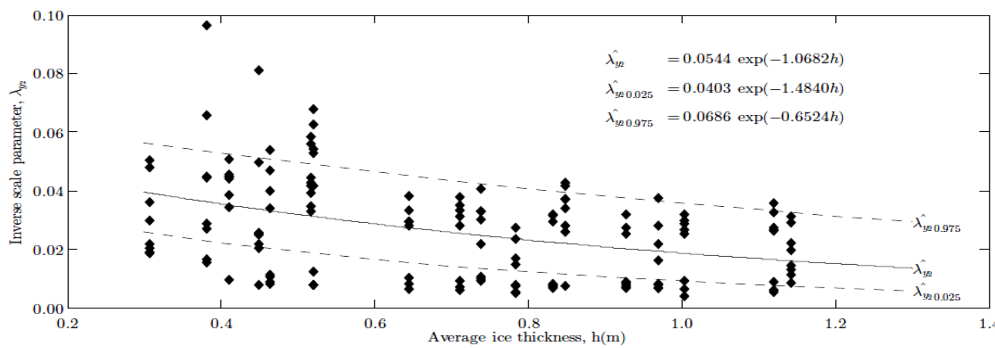
Conditional distribution for a given stationary condition

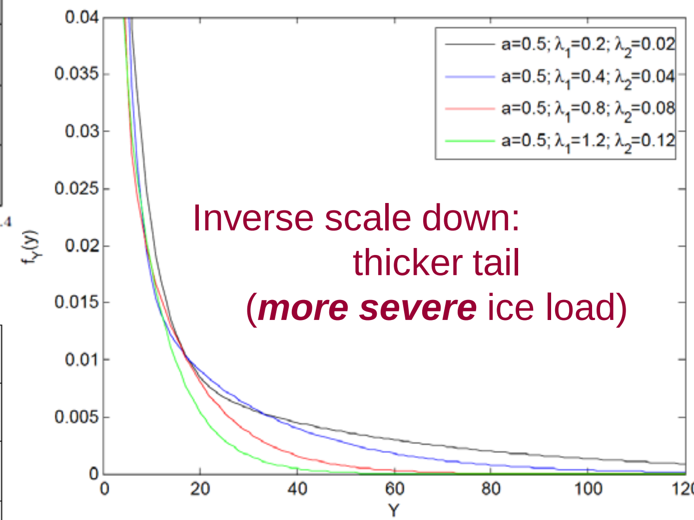
3-EXP

$$F_{Y|H_i}(y|h) = a(h)\left\{1-\exp(-\lambda_{y_1}(h)\,y)\right\} + (1-a(h))\left\{1-\exp(-\lambda_{y_2}(h)\,y)\right\}$$









Inverse scale down:
thicker tail
(***more severe*** ice load)

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Bayesian updating for three different probability models:

- Exponential distribution
- Exponential distribution, upper tail data
- Weibull distribution
- Combinations of these models are also addressed
- Probability of failure is computed for a fixed threshold
- Simplistic data set is applied

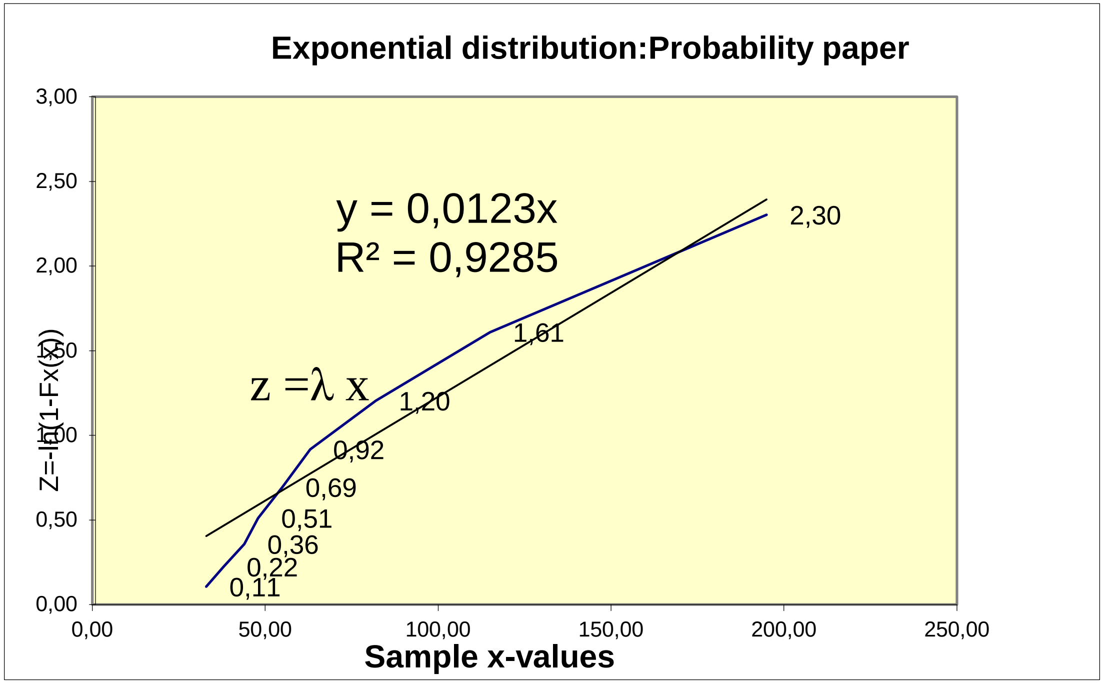
Peak ice load data set and exponential model

Simplistic data set (9 samples):

x=[33, 38, 44, 48, 55, 63, 82, 115, 195]

Exponential pdf:

$$f_X(x) = \lambda \exp(-\lambda x)$$



Prior pdf of parameter λ and likelihood function

Prior pdf of parameter is taken to be uniform between upper and lower limits:

$$\pi(\boldsymbol{\theta}) = \pi(\lambda) = \frac{1}{(\lambda_{\max} - \lambda_{\min})}$$

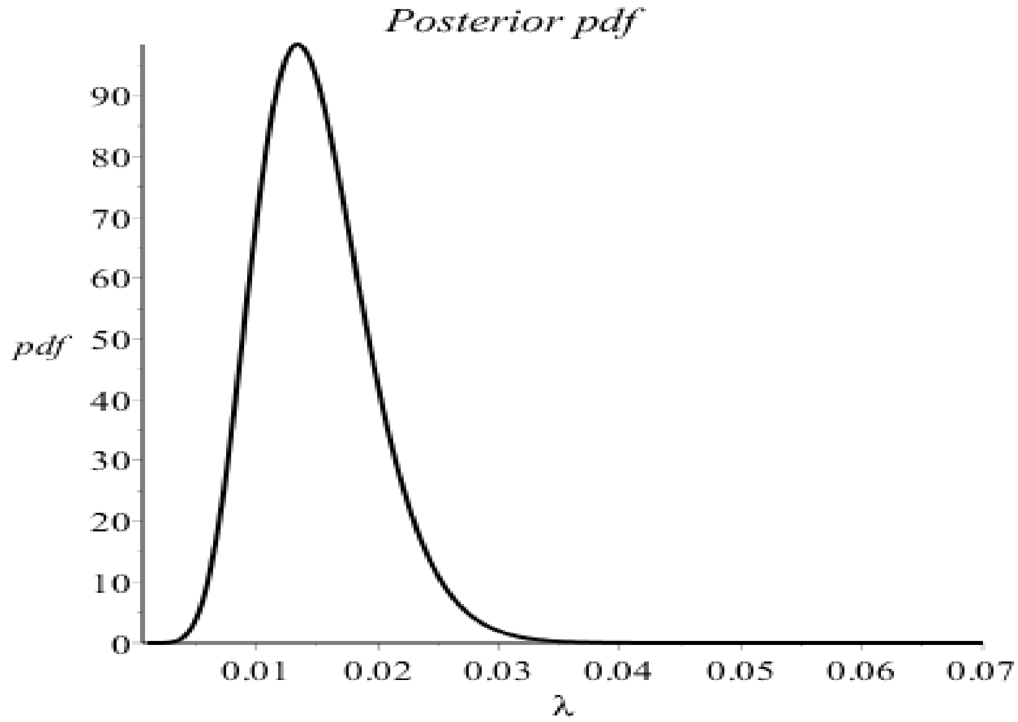
Likelihood function:

$$f_{\mathbf{X}|\Lambda}(\mathbf{x} \mid \lambda) = \lambda \exp(-\lambda x_1) \cdot \lambda \exp(-\lambda x_2) \cdots \lambda \exp(-\lambda x_n) = \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)$$

Bayesian updating of parameter λ:

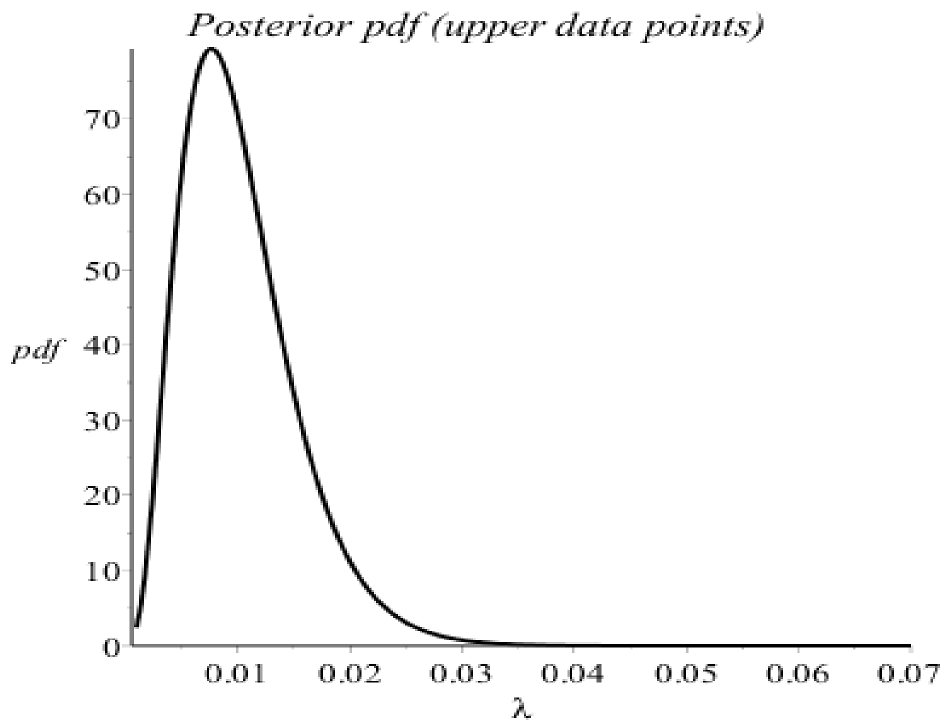
$$f_{\Lambda|\mathbf{X}}(\lambda \mid \mathbf{x}) = \frac{\lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)\left\{\frac{1}{(\lambda_{\max} - \lambda_{\min})}\right\}}{\int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)\left\{\frac{1}{(\lambda_{\max} - \lambda_{\min})}\right\} d\lambda} = \frac{\lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)}{\int_{\lambda_{\min}}^{\lambda_{\max}} \lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right) d\lambda} = C\lambda^n \exp\left(-\lambda \sum_{i=1}^{n} x_i\right)$$

Posterior pdf of parameter λ for the given sample



Normalization constant   C = 4.5e-23

Posterior pdf of parameter λ for «upper tail» of the given sample, i.e. including only largest three values



Normalization constant C = 2.8e-10

Weibull model: Prior pdfs of parameters (s, η) and likelihood function

Weibull pdf:

$$f_{\mathrm{X}}(\mathrm{x}) = \eta\left(s^{-\eta}\right)x^{\eta-1}\exp\left(-\left\{\frac{x}{s}\right\}^{\eta}\right)$$

Prior pdfs uniform between upper and lower limits:

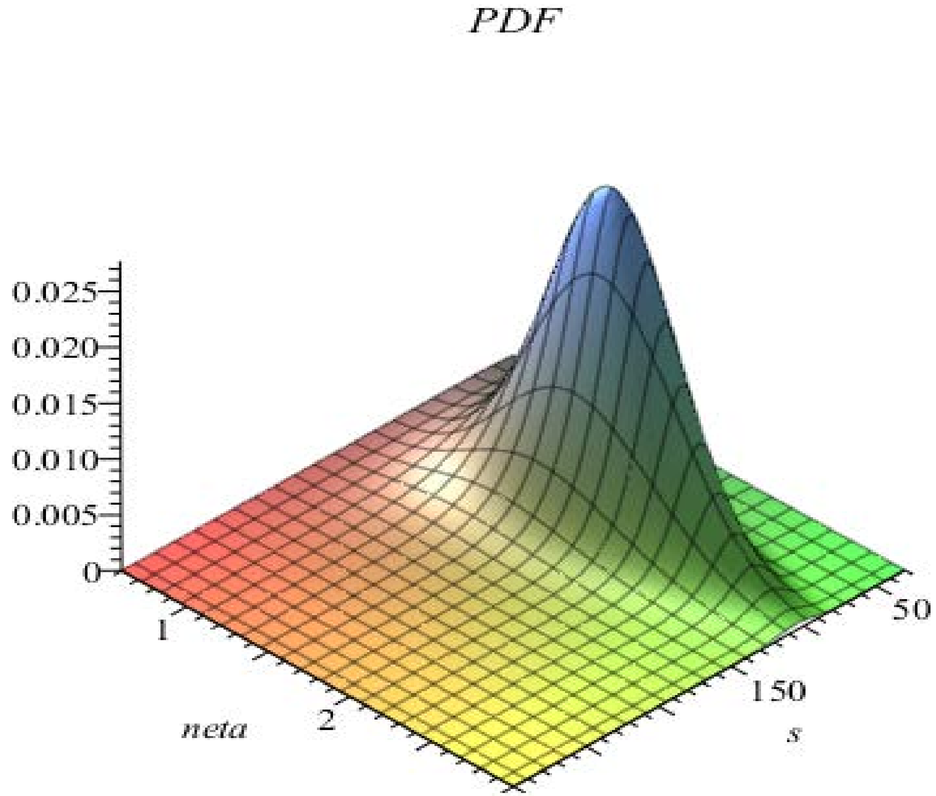
$$\pi(s,\eta) = \frac{1}{\left(\eta_{\mathrm{max}} - \eta_{\mathrm{min}}\right)} \cdot \frac{1}{\left(s_{\mathrm{max}} - s_{\mathrm{min}}\right)}$$

Likelihood function:

$$f_{\mathbf{X}|S,\mathrm{H}}(\mathbf{x} \mid s,\eta) = \eta\left(s^{-\eta}\right)x^{\eta-1}\exp\left(-\left\{\frac{x}{s}\right\}^{\eta}\right) \cdot \eta\left(s^{-\eta}\right)x^{\eta-1}\exp\left(-\left\{\frac{x}{s}\right\}^{\eta}\right) \cdots \eta\left(s^{-\eta}\right)x^{\eta-1}\exp\left(-\left\{\frac{x}{s}\right\}^{\eta}\right)$$

$$= \eta^{n}\left(s^{-n\eta}\right)\left(\prod_{i-1}^{n} x^{n(\eta-1)}\right)\exp\left(-\left(s^{-\eta}\right)\sum_{i=1}^{n} x_i^{\eta}\right)$$

Posterior pdf of parameters (s, η) for the given sample



Limits of parameters: s = 30-300, = 0.6-3.0
Normalization constant C = 0.38e-18

Extreme value distribution based on posterior distributions for given threshold $x_{cap} = 12.5 \cdot \lambda_{\min}$

Exponential:

$$p_f\left(x_{cap}, \lambda\right) = 1 - F_{X_e|\lambda}\left(\mathrm{x}_{\mathrm{cap}} \mid \lambda\right) = 1 - \left[1 - \exp\left(-\lambda x_{cap}\right)\right]^N$$

$$\approx N \cdot \exp(-\lambda \cdot \mathrm{x}_{\mathrm{cap}})$$

Weibull:

$$p_f\left(x_{cap}, s, \eta\right) = 1 - F_{X_e|\lambda}\left(\mathrm{x}_{\mathrm{cap}} \mid \lambda\right) = 1 - \left[1 - \exp\left(-\left\{\frac{x_{cap}}{s}\right\}^{\eta}\right)\right]^N \approx N \exp\left(-\left\{\frac{x_{cap}}{s}\right\}^{\eta}\right)$$

Conditional probability of failure for posterior distributions

Exponential:

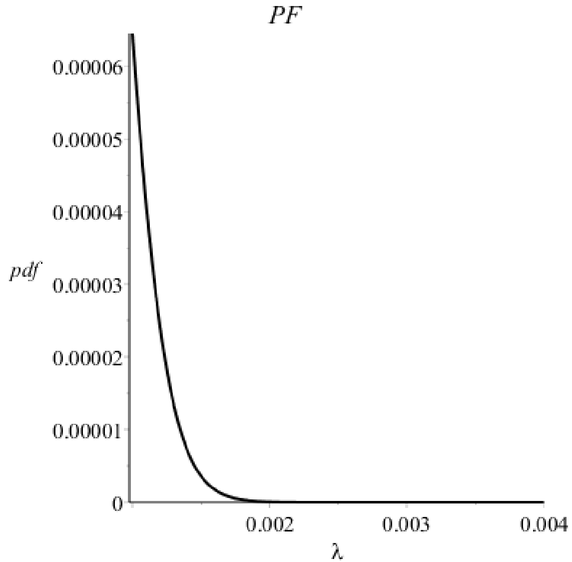
$$p_f\left(x_{cap},\lambda\right)=1-F_{X_e|\lambda}\left(\mathrm{x}_{\mathrm{cap}}\mid\lambda\right)=1-\left[1-\exp\left(-\lambda x_{cap}\right)\right]^N\approx N\exp(-\lambda x_{cap})$$

Weibull:

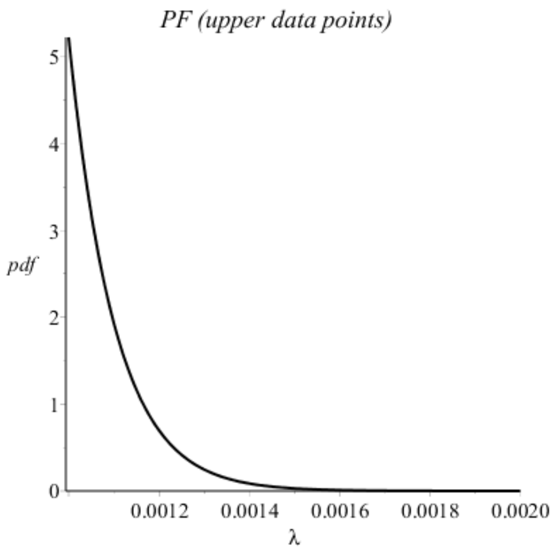
$$p_f\left(x_{cap},s,\eta\right)=1-F_{X_e|\lambda}\left(\mathrm{x}_{\mathrm{cap}}\mid\lambda\right)=1-\left[1-\exp\left(-\left\{\frac{x_{cap}}{s}\right\}^{\eta}\right)\right]^N\approx N\exp\left(-\left\{\frac{x_{cap}}{s}\right\}^{\eta}\right)$$

Graphs of conditional probability of failure based on posterior distributions, exponential models

Exponential:

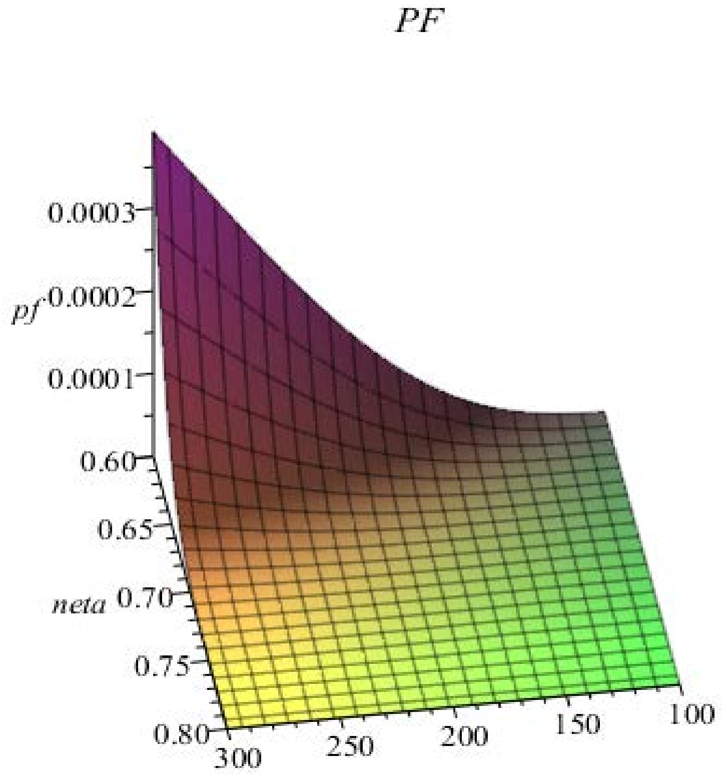


Exponential upper tail



Conditional probability of failure for posterior distribution, Weibull model

Weibull:



Unconditional probability of failure for posterior distributions

Exponential:

$$p_f\left(x_{cap}\right)=\int_{\lambda_{\min}}^{\lambda_{\max}} p_f\left(x_{cap},\lambda\right)\cdot f_{\Lambda|\mathbf{x}}\left(\lambda\,|\,\mathbf{x}\right)d\lambda = 3.2e-8$$

Exponential «upper tail»

$$p_f\left(x_{cap}\right)=\int_{\lambda_{\min}}^{\lambda_{\max}} p_f\left(x_{cap},\lambda\right)\cdot f_{\Lambda|\mathbf{x}}\left(\lambda\,|\,\mathbf{x}\right)d\lambda = 3.1e-3$$

Weibull:

$$p_f\left(x_{cap}\right)\approx N\cdot C\int_{\eta_{\min}}^{\eta_{\max}}\int_{s_{\min}}^{s_{\max}} \eta^n\left(s^{-n\eta}\right)\left(\prod_{i=1}^{n} x_i^{(\eta-1)}\right)\exp\left(-\left(s^{-\eta}\right)\left\{\left(x_{cap}^{\eta}\right)+\sum_{i=1}^{n} x_i^{\eta}\right\}\right)ds\,d\eta$$

$$=1.0e-3$$

Combination of different probability models based on Bayesian updating

Bayesian parameter updating:

$$f_{\boldsymbol{\theta}|\mathbf{X}}(\boldsymbol{\theta}\,|\,\mathbf{x}) = \frac{f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}\,|\,\boldsymbol{\theta})\cdot\pi(\boldsymbol{\theta})}{\int_{\theta} f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}\,|\,\boldsymbol{\theta})\cdot\pi(\boldsymbol{\theta})\,d\boldsymbol{\theta}}$$

Bayesian updating for combination of probabilistic models, Mj:

$$f_{\boldsymbol{\theta},M_j|\mathbf{X}}(\boldsymbol{\theta},M_j\,|\,\mathbf{x}) = \frac{f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}\,|\,\boldsymbol{\theta},M_j)\cdot\pi(\boldsymbol{\theta}\,|\,M_j)\pi(M_j)}{\sum_{J=1}^{m}\int_{\theta} f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}\,|\,\boldsymbol{\theta},M_j)\cdot\pi(\boldsymbol{\theta}\,|\,M_j)\,d\boldsymbol{\theta}\,\pi(M_j)}$$

Combination of different probability models based on Bayesian updating

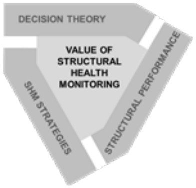
Continuous «model universe»:

$$f_{\boldsymbol{\theta},M|\mathbf{X}}(\boldsymbol{\theta},m\,|\,\mathbf{x})=\frac{f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}\,|\,\boldsymbol{\theta},m)\cdot\pi(\boldsymbol{\theta}\,|\,m)\pi(m)}{\int\limits_{m}\int\limits_{\theta} f_{\mathbf{X}|\boldsymbol{\theta}}(\mathbf{x}\,|\,\boldsymbol{\theta},m)\cdot\pi(\boldsymbol{\theta}\,|\,m)d\boldsymbol{\theta}\,\pi(m)dm}$$

Marginal posterior probability of probabilistic models:

$$P_{M_j|\mathbf{X}}(M_j\,|\,\mathbf{x})=\int\limits_{\theta} f_{\theta,M_j|\mathbf{X}}(\boldsymbol{\theta},M_j\,|\,\mathbf{x})d\boldsymbol{\theta}$$

$$f_{M|\mathbf{X}}(m\,|\,\mathbf{x})=\int\limits_{\theta} f_{\theta,M|\mathbf{X}}(\boldsymbol{\theta},m\,|\,\mathbf{x})d\boldsymbol{\theta}$$



Combination of the three different probability models for the example data set:

Prior probabilities for combination of all three distributions:

$$P_{Exp} = 0.3 \qquad P_{Exp\ tail} = 0.3 \qquad P_{Weib} = 0.3$$

Corresponding posterior probabilities:

$$P_{Exp} = 0. \qquad P_{Exp\ tail} = 1.0 \qquad P_{Weib} = 0.$$

Prior probabilities for combination of exponential and Weibull models:

$$P_{Exp} = 0.5 \qquad P_{Weib} = 0.5$$

Corresponding posterior probabilities:

$$P_{Exp} = 0. \qquad P_{Weib} = 1.0$$

Prior probabilities for combination of exponential «upper tail» and Weibull models:

$$P_{Exp\ tail} = 0.5 \qquad P_{Weib} = 0.5$$

Corresponding posterior probabilities:

$$P_{Exp\ tail} = 1.0 \qquad P_{Weib} = 0.$$

Resulting failure probabilities for the different combinations of models (example Exponential and Weibull):

$$p_f\left(x_{cap}\right) = P\left(M_{Exp}\right)C^{-1}{}_{Exp}\int_{\lambda_{\min}}^{\lambda_{\max}} N\cdot\lambda^n \exp\left(-\lambda\left[x_{cap}+\sum_{i=1}^{n}x_i\right]\right)d\lambda$$

$$+P(M_{Weibull})\cdot N\cdot C^{-1}{}_{Weib}\int_{\eta_{\min}}^{\eta_{\max}}\int_{s_{\min}}^{s_{\max}}\eta^n\left(s^{-n\eta}\right)x^{n(\eta-1)}\exp\left(-\left(s^{-\eta}\right)\left\{\left(x_{cap}^{\eta}\right)+\sum_{i=1}^{n}x_i^{\eta}\right\}\right)ds\,d\eta$$

Failure probability for only exponential distribution: 3.2e-8

Failure probability for combination of all three distributions: 3.1e-3

Failure probability for combination of exponential and Weibull models: 1.0e-3

Failure probability for combination of exponential «upper tail» and Weibull models: 3.1e-3

Summary of results

Prior and posterior probabilities :

Prior

$P_{Exp} = 0.3 \quad P_{Exp\ tail} = 0.3 \quad P_{Weib} = 0.3$

$P_{Exp} = 0.5 \quad P_{Weib} = 0.5$

$P_{Exp\ tail} = 0.5 \quad P_{Weib} = 0.5$

Posterior

$P_{Exp} = 0. \quad P_{Exp\ tail} = 1.0 \quad P_{Weib} = 0.$

$P_{Exp} = 0. \quad P_{Weib} = 1.0$

$P_{Exp\ tail} = 1.0 \quad P_{Weib} = 0.$

Failure probability (per year) for individual models and for the different combinations of models:

Exponential:	Exponential-uppertail	Weibull
3.2e-8	3.1e-3	1.0e-3

All three models: 3.1e-3

Exponential and Weibull: 1.0e-3

Exponential-uppertail and Weibull: 3.1e-3

Conclusions

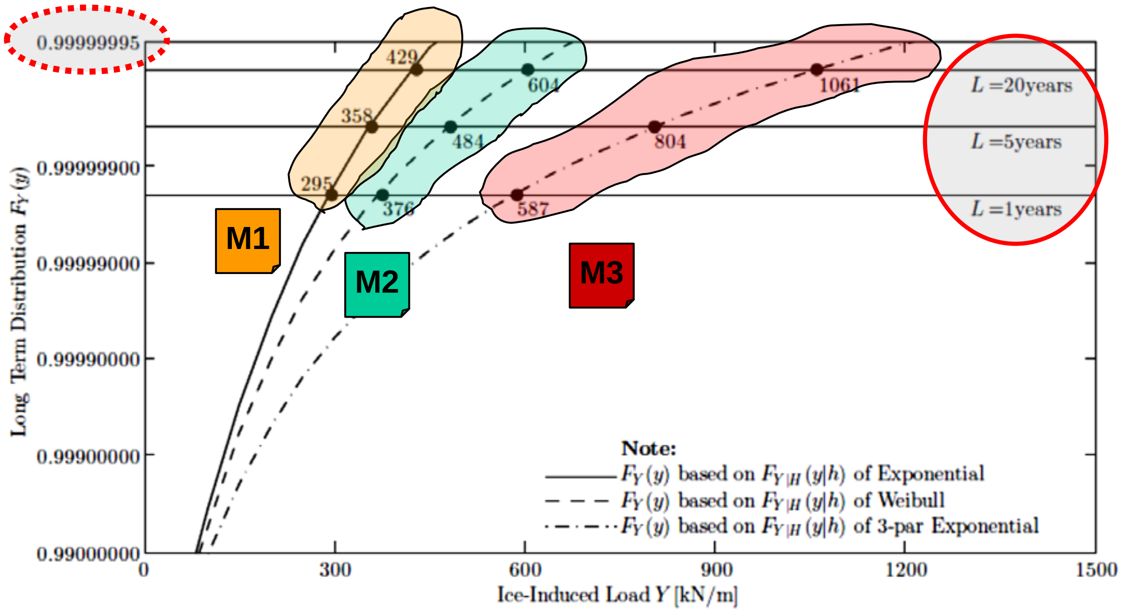
- Probabilistic models for peak ice loads acting on a ship in arctic areas are considered
- Bayesian updating for a simplistic data set is considered
- Annual probability of failure for a fixed capacity threshold is computed for different models (based on posterior distribution functions)
- Mixture of different models which are members of the chosen “model universe” are assessed by means of posterior probabilities for the different members
- For present data set, there is a strong ranking the three models based on the posterior probability (binary-type probabilities)

NTNU

Long term statistics: Comparison of extreme values for the three relevant models

Comparison of 3 models:

1. Exp
2. Weibull
3. 3-Exp



(a) all peak amplitude approach.

CONCLUDING REMARKS:

- ✓ *Initial/parent distribution of the ice load peak process*
- ✓ *Bayesian updating of parameters for different models*
- ✓ *Bayesian updating of the «plausibilities/credibilities» of the different models*
- ✓ *Computation of posterior failure probabilities for a fixed threshold value of the peak ice load*

Credits: A. Renner (blogs.esa.int)