Probabilistic Modelling

- Maximum Likelihood Method
- Bayesian regression analysis
- Determination of model uncertainties

Professor John Dalsgaard Sørensen
Department of Civil Engineering
Aalborg University, Denmark
jds@civil.aau.dk

Litterature

- Faber, M.H.: Faber MH. Statistics and Probability Theory in Pursuit of Engineering Decision Support: Springer; 2012. ISBN 978-94-007-4055-6
  - Pp 85-104
- EN 1990: Basis of structural design. 2002.
  - Annex D
- JCSS PMC – part 3-9: Model uncertainties
- ISO 2394: 2015. General principles on reliability for structures.

Probabilistic Modelling

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ESTIMATION OF DISTRIBUTION PARAMETERS

- Maximum Likelihood method
- Moment method
- Least squares method
- Bayesian statistics
- …

Maximum Likelihood method

Likelihood function: probability that the actual data are outcomes of a given distribution with given statistical parameters.

It is assumed that the data are statistically independent!

Example: Weibull distribution

Distribution function:

$$F(x) = 1 - \exp\left(-\left(\frac{x-\tau}{w-\tau} \right)^k \right)$$

density function

$$f(x) = \frac{k}{w-\tau} \left(\frac{x-\tau}{w-\tau} \right)^{k-1} \exp\left(-\left(\frac{x-\tau}{w-\tau} \right)^k \right)$$

Likelihood function:

$$L(k,w,\tau) = \prod_{i=1}^{n} f(x_i | k,w,\tau)$$

The Log-Likelihood-function becomes:

$$\ln L(k,w,\tau) = \ln\left(\prod_{i=1}^{n} f(x_i | k,w,\tau) \right) = \sum_{i=1}^{n} \ln\left(\frac{k}{w-\tau} \left(\frac{x_i-\tau}{w-\tau} \right)^{k-1} \exp\left(-\left(\frac{x_i-\tau}{w-\tau} \right)^k \right) \right)$$

Statistical parameters: k, w, τ

$x_i \quad : i = 1, ..., n \qquad$ data values

Optimal parameters k, w, τ determined from optimization problem:

$$\max_{k,w,\tau} \ln L(k, w, \tau)$$

Statistical uncertainty: if the number of data is larger than 25-30 k, w, τ can be assumed asymptotically normal distributed with expected values equal to the solution of the optimization problem and with covariance matrix:

$$\mathbf{C}_{k,w,\tau} = -\left[\mathbf{H}_{k,w,\tau}\right]^{-1} = \begin{bmatrix} \sigma_k^2 & \rho_{kw}\sigma_k\sigma_k & \rho_{k\tau}\sigma_k\sigma_\tau \\ \rho_{kw}\sigma_k\sigma_w & \sigma_w^2 & \rho_{w\tau}\sigma_w\sigma_\tau \\ \rho_{k\tau}\sigma_k\sigma_\tau & \rho_{w\tau}\sigma_w\sigma_\tau & \sigma_\tau^2 \end{bmatrix}$$

where $\mathbf{H}_{k,w,\tau}$ is the Hessian matrix with second derivatives of the Log-Likelihood function.

Example
Wind data – Peak Over Threshold (POT) method – fit by Weibull distribution

Procedure:

1. By POT the n largest wind speeds are identified from the data set over T years.
   It is required that two consecutive data has a distance on at least 1 day

2. A distribution function is fitted to the data using the Maximum Likelihood Method.
   Here the 3-parameter Weibull distribution is used.

3. The wind speed corresponding to a given return period is estimated
   - Statistical uncertainty is not taken into account
   - Statistical uncertainty is taken into account

Step 1: Identification of data
n statistically independent (local maximum) data are identified over a period of T years: $x_i \quad : i = 1,\ldots,n$

Step 2: Fit of distribution function by Maximum Likelihood Method
A distribution function, $F(x)$ with corresponding density function, $f(x)$ is fitted to the data using the Maximum Likelihood Method.

Example: Weibull distribution

Density function:

$$f(x) = \frac{k}{w-\tau}\left(\frac{x-\tau}{w-\tau}\right)^{k-1} \exp\left(-\left(\frac{x-\tau}{w-\tau}\right)^{k}\right)$$

The Log-Likelihood-function becomes:

$$\ln L(k,w) = \ln\left(\prod_{i=1}^{n} f(x_i)\right) = \sum_{i=1}^{n} \ln\left(\frac{k}{w-\tau}\left(\frac{x_i-\tau}{w-\tau}\right)^{k-1} \exp\left(-\left(\frac{x_i-\tau}{w-\tau}\right)^{k}\right)\right)$$

It is assumed that the lower threshold τ is fixed (selected by a sensitivity analysis prior to fitting k,w). The optimal parameters $(\hat{k},\hat{w})$ are determined from optimization problem:

$$\max_{k,w} \ln L(k,w)$$

which can be solved using standard optimization algorithms.

Statistical uncertainty: k,w: Normal distributed with expected values equal to the solution of the optimization problem and covariance matrix:

$$\mathbf{C}_{k,w} = -\left[\mathbf{H}_{k,w}\right]^{-1} = \begin{bmatrix} \sigma_k^2 & \rho_{k,w}\sigma_k\sigma_w \\ \rho_{k,w}\sigma_k\sigma_w & \sigma_w^2 \end{bmatrix}$$

where $\mathbf{H}_{k,w}$ is the Hessian matrix with second derivatives of the Log-Likelihood function.

This means that the stochastic uncertainty is modelled by assuming k,w is Normal distributed with

- Mean values: $(\mu_k, \mu_w) = (\hat{k}, \hat{w})$
- Standard deviations: (σ_k, σ_w)
- Correlation coefficient: $\rho_{k,w}$

Step 3: Calculation of values with different return periods

The distribution for annual maximum value is obtained from

$$F_{\max}(x) = \left(F(x)\right)^{\lambda}$$

where the number of data per year is $\lambda = n / T$.

If the data are fitted to e.g. a Weibull distribution then:

$$F_{\max}(t) = \left(F(t)\right)^{\lambda} = \left(1 - \exp\left(-\left(\frac{t-\tau}{w-\tau}\right)^{k}\right)\right)^{\lambda}$$

The R-year value x_R is determined from

$$F_{\max}(x_R) = 1 - \frac{1}{R}$$

An approximate value of the R-year value can be obtained from

$$F(x_R) = 1 - \frac{1}{\lambda \cdot R}$$

If statistical uncertainty is not taken into account then

$$x_R = F^{-1}\left(1 - \frac{1}{\lambda \cdot R}\right)$$

where $F^{-1}(\ )$ is the inverse distribution function.

If statistical uncertainty is included then the R-year value can be obtained by calculating x_R from

$$P\left(x_R - F^{-1}\left(\Phi\left(U_1\right);k,w\right) \le 0\right) = 1 - \frac{1}{\lambda \cdot R}$$

where $F^{-1}\left(\Phi\left(U_1\right);k,w\right)$ is the inverse distribution function corresponding to the probability $\Phi\left(U_1\right)$. U_1 is a standard Normal distributed stochastic variable with mean value = 0 and standard deviation = 1.

$g = x_R - F^{-1}\left(\Phi\left(U_1\right);k,w\right)$ can be considered as a limit state equation with three stochastic variables: $\left(U_1,k,w\right)$ where $\left(k,w\right)$ are Normal distributed with mean values $\left(\mu_k,\mu_w\right)$ standard deviations, $\left(\sigma_k,\sigma_w\right)$ and correlation coefficient $\rho_{k,w}$.

Using the reliability index technique (FORM) then x_R can be obtained by iteration.

Example:

$n = 38$ wind speed data [m/s] over $T = 7$ years:

24.00
24.00
24.09
24.14
24.14
24.19
24.20
24.20
24.28
24.45
24.49
24.69
24.71
24.71
24.79
24.83
25.16
25.26
25.26
25.49
25.49
25.49
25.69
25.69
25.79
26.19
26.19
26.19
26.49
26.50
27.40
27.80
27.80
27.99
28.15
28.15
28.30
29.40

STATREL (from rcp) is used:

* Parameter Estimation *
Selected estimation method: Maximum likelihood
(using Schittkowski-algorithm)
Selected stochastic model: Weibull (min) (9)

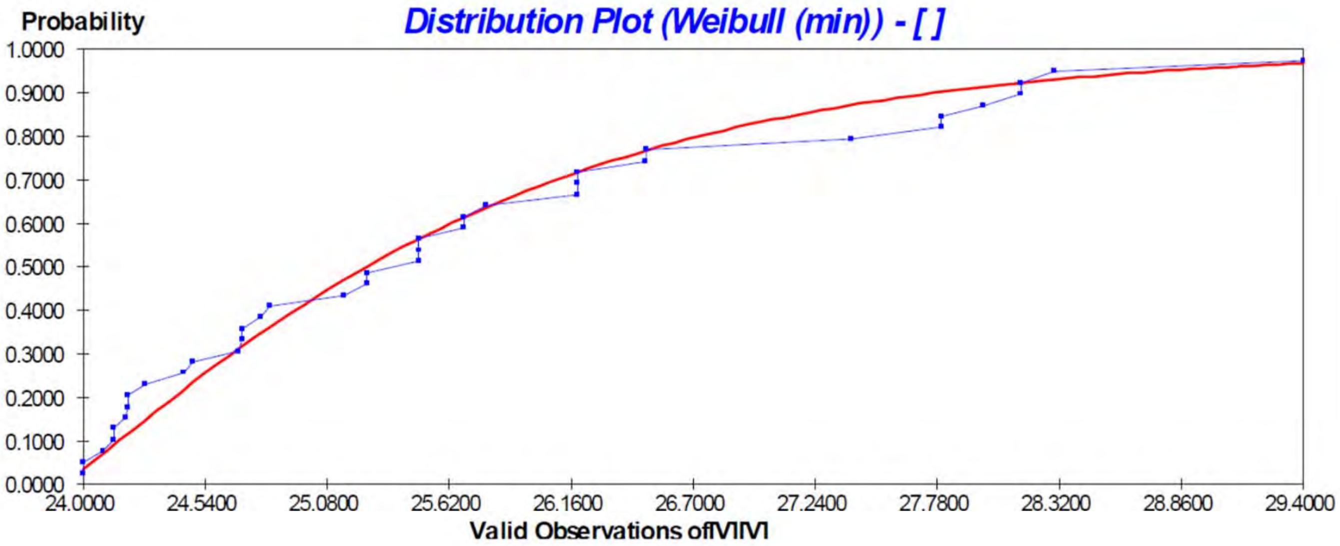
Parameter 1 :	w	25.7661	
Parameter 2 :	k	1.14575	
Parameter 3 :	tau	23.9000	(known on input)

Standard deviations for estimated parameters:

Parameter 1 :	.278352
Parameter 2 :	.149704
Parameter 3 :	(known on input)

Matrix of correlation coefficients for parameters:

1	1.00000	.313125	.000000
2	.313125	1.00000	.000000
3	.000000	.000000	.000000



Characteristic values

Return period [years]	Without statistical uncertainty [m/s]	With statistical uncertainty [m/s]
20	31.1	31.7
50	32.3	33.4
100	33.2	34.8

Matlab:

Example: Fit to Gumbel ditribution:

$$F_X\left(x\,|\,\alpha,\beta\right)=\exp\left(-\exp\left(-\alpha\left(x-\beta\right)\right)\right)$$

$$f_X\left(x\,|\,\alpha,\beta\right)=\alpha\exp\left(-\alpha\left(x-\beta\right)\right)\exp\left(-\exp\left(-\alpha\left(x-\beta\right)\right)\right)-$$

```
% Load of data
Data = [dlmread('wind_speeds_peaks.txt')];

% Calculation of descriptive statistical parameters
Data_mean = mean(Data)
Data_std  = std(Data)

% Definition of LogLikelihood Function
fun = @(x) -sum(log(x(1).*exp(-x(1).*(Data-x(2))).*exp(-exp(-x(1).*(Data-x(2))))));

% Initial Parameters
x0  = [1,1];

% Minimization of the Negative LogLikelihood Function
[x,fval,exitflag,output,grad,hessian] = fminunc(fun,x0)

% Estimation of Correlation matrix (Minus Removed due to Negative LogLikehood Function)
C   = inv(hessian)

% Estimation of Standard Deviations and Correlations
S1  = sqrt(C(1,1))
S2  = sqrt(C(2,2))
R12 = C(1,2)/(S1*S2)
```

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Bayesian regression analysis

n-dimensional linear regression model

$$y(\boldsymbol{\beta}) = \sum_{j=1}^{n} x_j \beta_j + \varepsilon$$

$\mathbf{x} = \left(x_1, x_2, \ldots, x_n\right)$ deterministic (known) quantities

$\boldsymbol{\beta} = \left(\beta_1, \beta_2, \ldots, \beta_n\right)$ (unknown) regression parameters

ε Normal distributed stochastic variable modelling lack of fit
mean value = 0 and standard deviation = $1/\sqrt{h}$

N observations $\hat{\mathbf{y}} = (\hat{y}_1, \hat{y}_2, \ldots, \hat{y}_N)$ and x_{ji}, $j = 1, \ldots, n$; $i = 1, \ldots, N$ are available

Likelihood function corresponding to the observed outcomes:

$$f_{\mathbf{y}}(\hat{\mathbf{y}} | \boldsymbol{\beta}, h) = \prod_{i=1}^{N} f_{\varepsilon}(\hat{y}_i - \sum_{j=1}^{n} x_{ji} \beta_j | h)$$

Maximum Likelihood Method used for obtaining estimate $\hat{\mathbf{x}}$ of regression

The following quantities are determined

$$\boldsymbol{\eta} = \mathbf{x}^T \mathbf{x} \qquad\qquad (n \times n \text{ matrix})$$

$$p = \text{ rank } (\mathbf{x})$$

$$\nu = N - p$$

$$\mathrm{v} = \frac{1}{\nu}(\hat{\mathbf{y}} - \mathbf{x}\hat{\boldsymbol{\beta}})^T (\hat{\mathbf{y}} - \mathbf{x}\hat{\boldsymbol{\beta}})$$

where $\quad \hat{\boldsymbol{\beta}} = \boldsymbol{\eta}^{-1} \mathbf{x}^T \hat{\mathbf{y}}$

Since the number of experiments is often limited the regression parameters $\hat{\boldsymbol{\beta}}$ and the precision h cannot be determined with certainty, and therefore $\hat{\boldsymbol{\beta}}$ and h are considered as outcomes of stochastic variables $\boldsymbol{\beta}$ and H.

If prior knowledge in the form of n^{pr} observations $(\hat{y}_1^{pr}, \mathbf{x}_1^{pr}), (\hat{y}_2^{pr}, \mathbf{x}_2^{pr})$, ..., $(\hat{y}_{n^{pr}}^{pr}, \mathbf{x}_{n^{pr}}^{pr})$ of y for given experimental design are available the corresponding prior parameters η', p', ν' and v' can be determined from above equations

If prior knowledge is available in the form of subjective information on the expected values $E[\boldsymbol{\beta}], E[h]$ of $\boldsymbol{\beta}$ and h, covariance matrix $\mathbf{C}_{\boldsymbol{\beta}}$ of $\boldsymbol{\beta}$ and standard deviation σ_h of h, the prior parameters $\boldsymbol{\eta}', \hat{\boldsymbol{\beta}}', \nu'$ and v' can be obtained from

$$\mathrm{v}' = \frac{1}{E[h]} \qquad \nu' = 2\left(\frac{E[h)}{\sigma_h}\right)^2$$

$$\hat{\boldsymbol{\beta}}' = E[\boldsymbol{\beta}] \qquad \boldsymbol{\eta}' = \mathrm{v}'\mathbf{C}_{\boldsymbol{\beta}}^{-1}$$

‘Convenient’ choice of prior distribution function: natural conjugate distributior

This distribution can be shown to be an N-dimensional Normal-Gamma distribution:

$$f_{\boldsymbol{\beta},h}(\boldsymbol{\beta},h \,|\, \hat{\boldsymbol{\beta}},\eta,\nu,\mathrm{v}) \propto$$

$$h^{\nu/2-1}\exp\left[-\frac{1}{2}h\nu\mathrm{v}\right]h^{p/2}\exp\left[-\frac{1}{2}h(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})^T\boldsymbol{\eta}(\boldsymbol{\beta}-\hat{\boldsymbol{\beta}})\right]$$

If the **prior distribution** for $(\boldsymbol{\beta},h)$ is assumed Normal-Gamma distributed with parameters (η', $\hat{\boldsymbol{\beta}}'$, ν' and v') and if an experiment gives the data set $(\hat{y}_1,\hat{\mathbf{x}}_1),(\hat{y}_2,\hat{\mathbf{x}}_2),\ldots,(\hat{y}_N,\hat{\mathbf{x}}_N)$ then the **posterior distribution** for $(\boldsymbol{\beta},h)$ will be Normal-Gamma with the parameters

$$\boldsymbol{\eta}'' = \boldsymbol{\eta}' + \boldsymbol{\eta} \qquad\qquad p'' = \mathrm{rank}(\eta'')$$

$$\hat{\boldsymbol{\beta}}'' = (\boldsymbol{\eta}'')^{-1}(\boldsymbol{\eta}'\hat{\boldsymbol{\beta}}' + \mathbf{x}^T\hat{\mathbf{y}})$$

$$\nu'' = (\nu' + p') + (\nu + p) - p''$$

$$\mathrm{v}'' = [(\nu'\mathrm{v}' + \hat{\boldsymbol{\beta}}'^T\boldsymbol{\eta}'\hat{\boldsymbol{\beta}}') + \hat{\mathbf{y}}^T\hat{\mathbf{y}} - \hat{\boldsymbol{\beta}}''^T\boldsymbol{\eta}''\hat{\boldsymbol{\beta}}'']$$

The posterior normal-gamma distribution is equivalent to the product of an N-dimensional Normal distribution (N) and a Gamma-2 distribution.

The (updated) **predictive distribution** of the response y given $\mathbf{x} = \left(x_1, x_2, \ldots, x_n\right)$ and given the observations $\hat{\mathbf{y}}$ is a Student-t distribution with density function

$$f_y(y|\hat{\mathbf{y}}) = \frac{\nu''^{\nu''/2}\Gamma(\frac{\nu''+1}{2})}{\sqrt{\pi}\Gamma(\nu''/2)}(\nu'' + (y - \mathbf{x}\hat{\boldsymbol{\beta}}'')^T(\eta_r / \mathrm{v}'')(y - \mathbf{x}\hat{\boldsymbol{\beta}}''))^{-\nu''/2-1/2}\sqrt{\eta_r/\mathrm{v}''}$$

where Γ is the Gamma function and

$$\eta_r = 1 - \mathbf{x}(\boldsymbol{\eta}'' + \mathbf{x}^T\mathbf{x})^{-1}\mathbf{x}^T$$

Note: if h (standard deviation) is assumed to be known the prior, posterior and predictive distributions simplifies to Normal distributions

Example: Modelling of fatigue data
SN-curve:

$$N = KS^{-m}$$

where K and m are material data to be determined

The SN-curve can also be written

$$\ln N = \ln K - m \ln S$$

This relation is linear in the n=2 regression parameters $(\beta_1, \beta_2) = (\ln K, m)$

The experimental design can be written $(x_1, x_2) = (1, \ln S)$

The experiment result is the number of cycles to failure, N, i.e. $y = \ln N$.

From laboratory tests a number of data sets $(\hat{y}, x_2) = (\ln N, \ln S)$ are obtained

The linear regression model can then be used and an updated (predictive) density function obtained for $y = \ln N$ taking into account the test data and subjective prior knowledge, if available

Example: steel yield strength (Faber p. 97-99)

Yield strength: N(mean value, standard deviation = 17.5 MPa)

Prior for mean value: N(350, 10) [MPa]

Data: 365, 347, 354, 362, 348 [MPa]

Posterior for mean value: N(353.22, 6.16) [MPa]

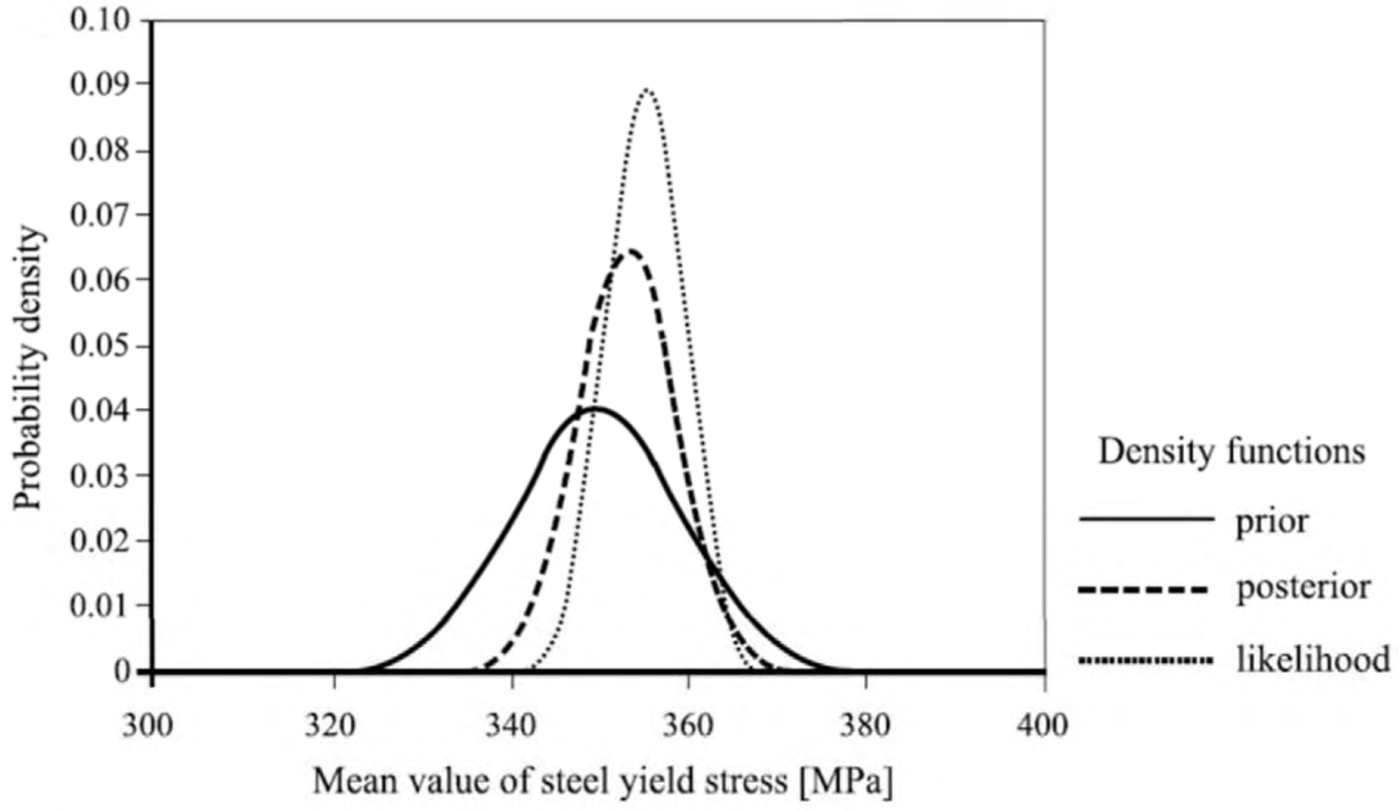


Fig. 5.6 Illustration of *prior* and *posterior* probability density and likelihood functions for the mean value of the steel yield stress

Predictive distribution for yield strength: N(353.22, 18.6) [MPa]

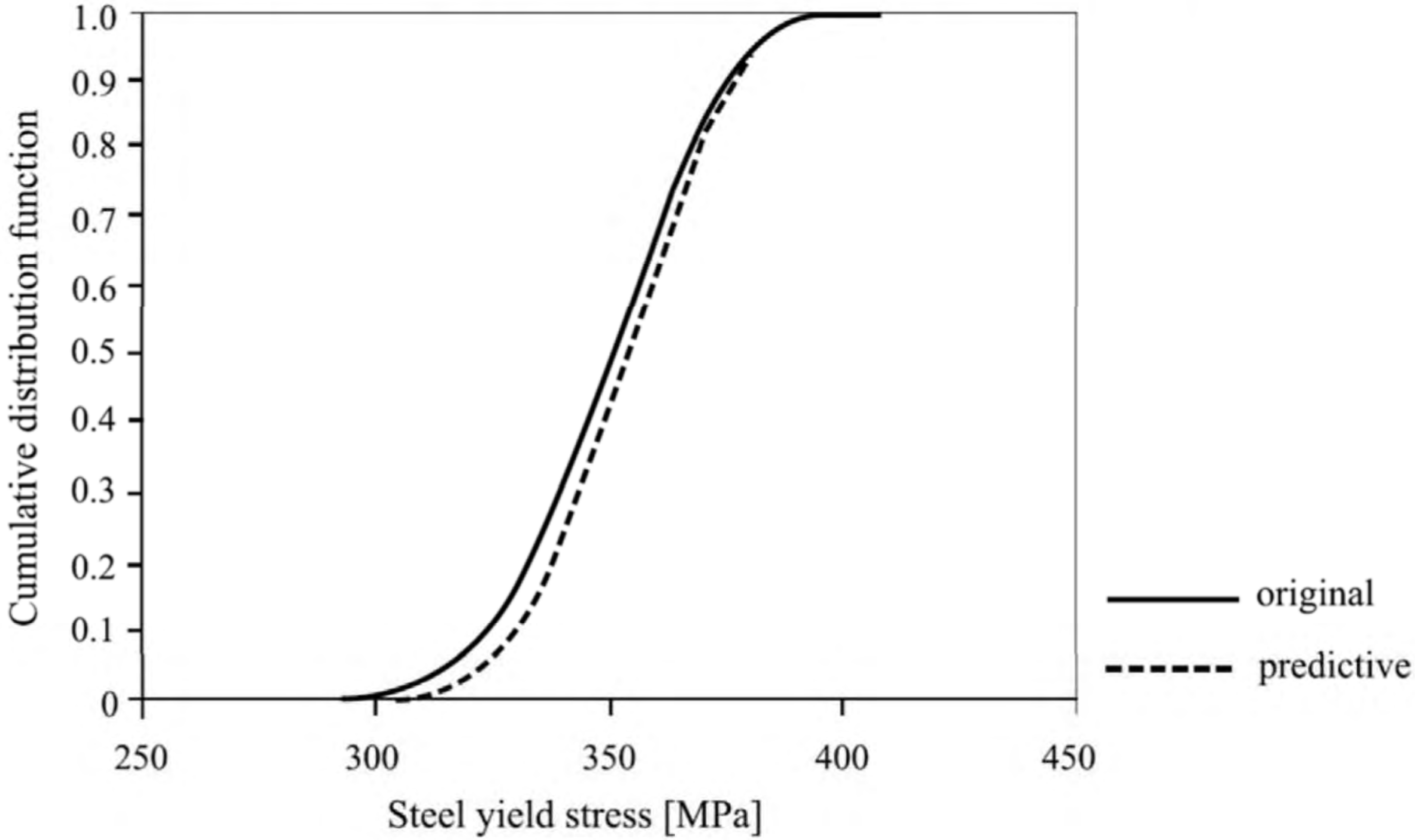


Fig. 5.7 Illustration of original and predictive probability distribution function for the steel yield stress

Example: tensile strength of timber (Faber p. 99-105)

X: Tensile strength

Y: tensile modulus of elasticity

Model:

$$Y = \beta_0 + \beta_1 X + \varepsilon$$

where $\varepsilon : \mathrm{N}(0, \sigma_\varepsilon)$

units in [MPa]

Data: measured values:

Tensile strength $\hat{\mathbf{x}}$ [MPa]	Tensile modulus of elasticity $\hat{\mathbf{y}}$ [MPa]
19.11	8426
12.30	7092
14.83	7347
13.18	7917

Least square method used to obtain (β_0, β_1) and σ_ε:

$(\beta_0, \beta_1) = (5274.12, 156.27)$ $\quad \sigma_\varepsilon = 446.25$

and covariance matrix

$$\mathbf{C}_{\boldsymbol{\beta}} = \sigma_\varepsilon^2 \mathbf{V}_\beta = \sigma_\varepsilon^2 \left(\hat{\mathbf{X}}^T \hat{\mathbf{X}}\right)^{-1} \mathbf{V}_\beta = 446.25^2 \begin{bmatrix} 8.29 & -0.54 \\ -0.54 & 0.004 \end{bmatrix} = \begin{bmatrix} 3700.4 & -241.6 \\ -241.6 & 16.3 \end{bmatrix}$$

Bayesian updating with additional measurements:

Tensile strength $\hat{\mathbf{x}}$ [MPa]	Tensile modulus of elasticity $\hat{\mathbf{y}}$ [MPa]
18.04	7581
24.10	9661

It is assumed that σ_ε is known

Prior parameters for (β_0, β_1):

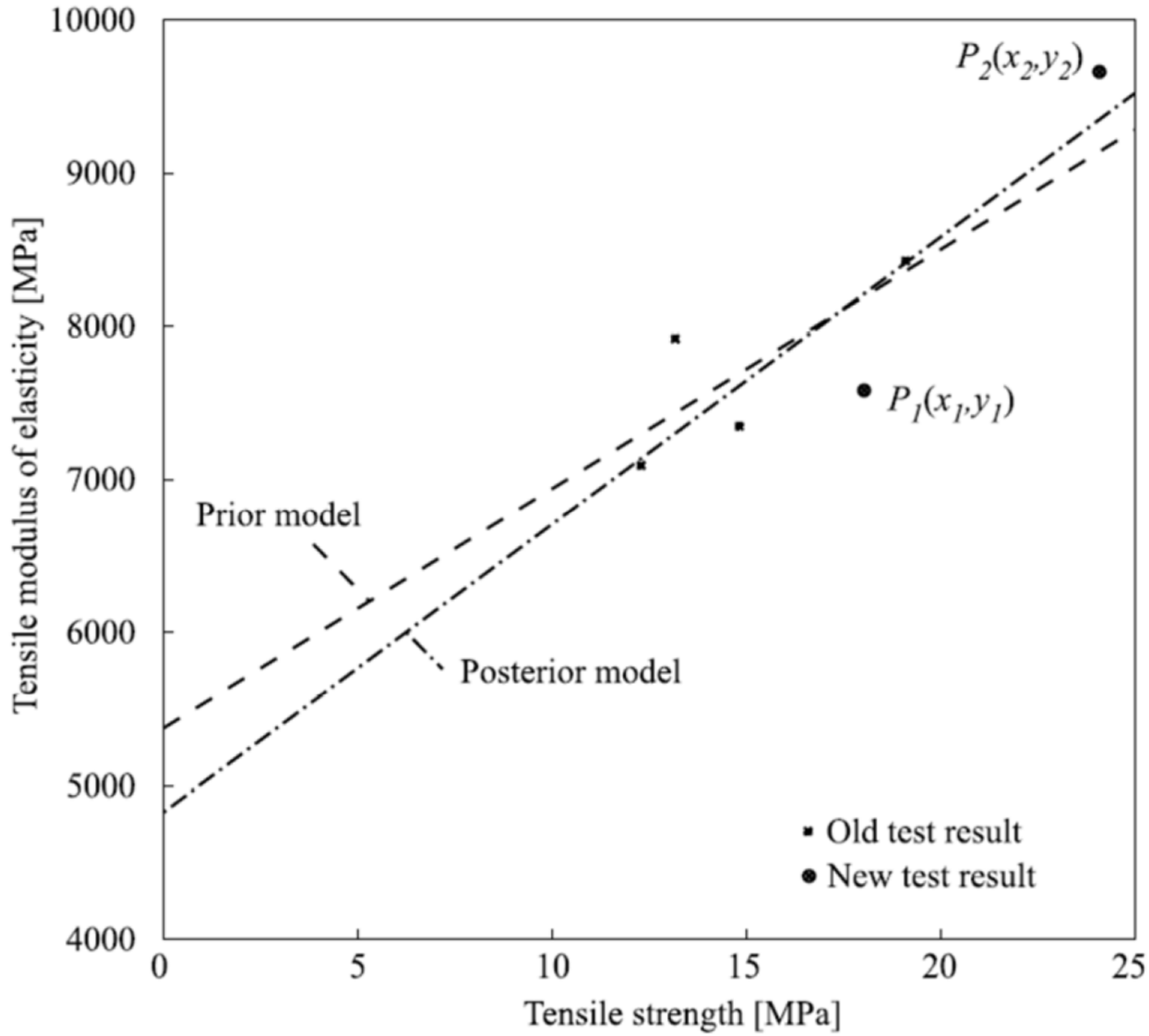
$$\boldsymbol{\beta}' = \begin{pmatrix} 5374.12 \\ 156.27 \end{pmatrix} \qquad \mathbf{V}'_\beta = \begin{pmatrix} 8.29 & -0.54 \\ -0.54 & 0.04 \end{pmatrix}$$

Posterior parameters for (β_0, β_1):

$$(\mathbf{V}''_\beta)^{-1} = \begin{pmatrix} 10.29 & 41.6 \\ 41.6 & 906.29 \end{pmatrix} \qquad \boldsymbol{\beta}'' = \begin{pmatrix} 4827.58 \\ 187.66 \end{pmatrix}$$

i.e. updated regression line:

$$Y = 4827.58 + 187.66X + \varepsilon$$


10000
9000
8000
7000
6000
5000
4000
Tensile modulus of elasticity [MPa]
0
5
10
15
20
25
Tensile strength [MPa]
$P_2(x_2,y_2)$
$P_1(x_1,y_1)$
Prior model
Posterior model
Old test result
New test result

Probabilistic Modelling

- Maximum Likelihood Method
- Bayesian regression analysis
- Determination of model uncertainties

Determination of model uncertainty

Types of uncertainty:

- **Physical uncertainty**
- **Measurement uncertainty**
- **Statistical uncertainty**: due to limited number of observations
- $\boxed{\textbf{Model uncertainty}}$

Aleatory

Epistemic

Estimation of model uncertainties

Model:

$$Y \cong b \cdot \Delta \cdot f_t \left(\mathbf{X}, \beta_1, \ldots, \beta_m\right)$$

f_t	mathematical / numerical / theoretical model
$\beta_1, \ldots, \beta_m$	parameters in theoretical model
$\mathbf{X}$	**physical uncertainties** – stochastic variables
b	bias in model
Δ	**model uncertainty**: LogNormal :LN(1, σ_Δ)

Available data from measurements or tests:

1	$\mathbf{x}_1$	$y_1 = f_{\exp,1}$
2	$\mathbf{x}_2$	$y_2 = f_{\exp,2}$
...	...	...
N	$\mathbf{x}_N$	$y_N = f_{\exp,N}$

In tests: realisations $\mathbf{x}_1, \ldots, \mathbf{x}_N$ of physical uncertainties $\mathbf{X}$ (i.e. known)

Assumption: test results are statistically independent

Estimation of model uncertainty parameters:

- Maximum Likelihood Method
- EN1990 method

Maximum likelihood method

$\beta_1,\ldots,\beta_m,b,\sigma_\Delta$ determined by Maximum Likelihood Method

Likelihood function:

$$L(R_1,\ldots,R_m,b,\sigma_\Delta)=\prod_{i=1}^{N} f_{\ln\Delta}(\ln f_{\exp,i}-\ln b-\ln f_t\left(x_i,R_1,\ldots,R_m\right)\big|\sigma_\Delta)$$

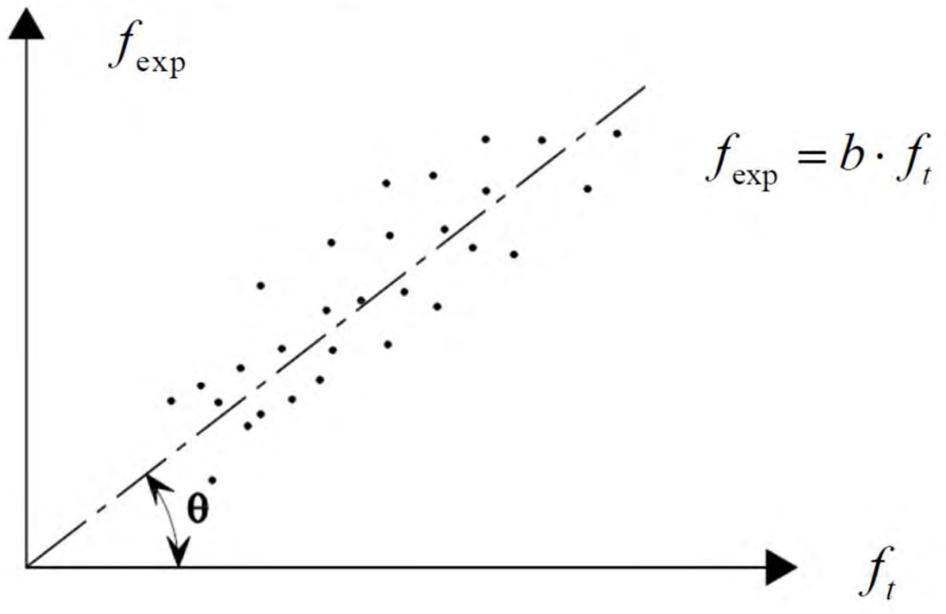
$f_{\ln\Delta}\left(r\middle|\sigma_\Delta\right)$ density function for $\ln\Delta$: Normal distribution

$R_1,\ldots,R_m,b,\sigma_\Delta$: solution to optimization problem:

$$\max_{R_1,\ldots,R_m,b,\sigma_\Delta} \ln L(R_1,\ldots,R_m,b,\sigma_\Delta)$$

In addition **statistical uncertainty** can be estimated if at least 25-30 sets are available

EN1990 method



Experimental results $f_{\exp}$ versus theoretical values f_t.

Model:

$$Y \cong b \cdot \Delta \cdot f_t(\mathbf{X})$$

b bias

Δ LN $(1, V_\Delta)$

Data:

Experiment: $f_{\exp,i} \quad ,i=1,2,...,N$ test data

Model: $f_{t,i} \quad ,i=1,2,...,N$ corresponding estimates by model

1) Calculate b (linear regression)

$$b=\frac{\sum_{i=1}^{N} f_{\exp,i}\cdot f_t(\mathbf{x}_i)}{\sum_{i=1}^{N} f_t(\mathbf{x}_i)^2}$$

2) Calculate:

$$\Delta_i=\ln\left(\frac{f_{\exp,i}}{b\cdot f_t(\mathbf{x}_i)}\right)$$

3) estimated mean value

$$\overline{\Delta}=\frac{1}{N}\sum_{i=1}^{N}\Delta_i$$

4) estimate of the standard deviation

$$s_\Delta=\sqrt{\frac{1}{N-1}\sum_{i=1}^{N}\left(\Delta_i-\overline{\Delta}\right)^2}$$

5) coefficient of variation of the model uncertainty:

$$V_\Delta=\sqrt{\exp\left(s_\Delta^2\right)-1}$$

Linearized model: $f_t \approx f_t(\boldsymbol{\mu}_{\mathbf{X}}) + \sum_{j=1}^{n} \left.\frac{\partial f_t}{\partial x_j}\right|_{\mathbf{X}=\boldsymbol{\mu}_{\mathbf{X}}} \left(x_j - \mu_{X_j}\right)$

6) expected value of model

$$\mu_Y = b \cdot \mu_t = b \cdot f_t\left(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_n}\right)$$

7) coefficient of variation of model due to the basic variables $\mathbf{X}$:

$$V_t^2 = \frac{1}{f_t\left(\mu_{X_1}, \mu_{X_2}, \dots, \mu_{X_n}\right)^2} \sum_{j=1}^{n} \left(\frac{\partial f_t}{\partial x_j} \sigma_{X_j}\right)^2$$

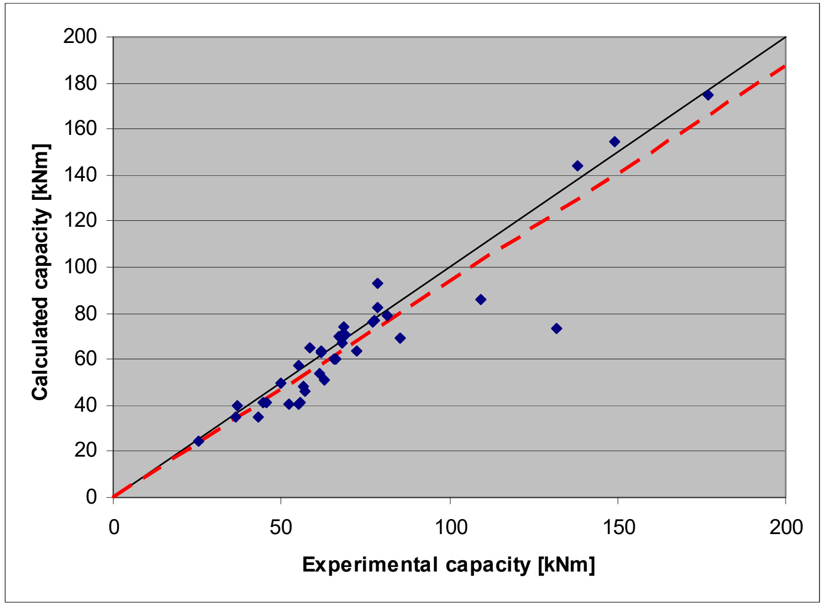
Total coefficient of variation of model:

$$\mathrm{V}_Y = \sqrt{V_\Delta^2 + V_t^2}$$

If the test results are subject to measurement uncertainty with coefficient of variation V_ε: Total coefficient of variation:

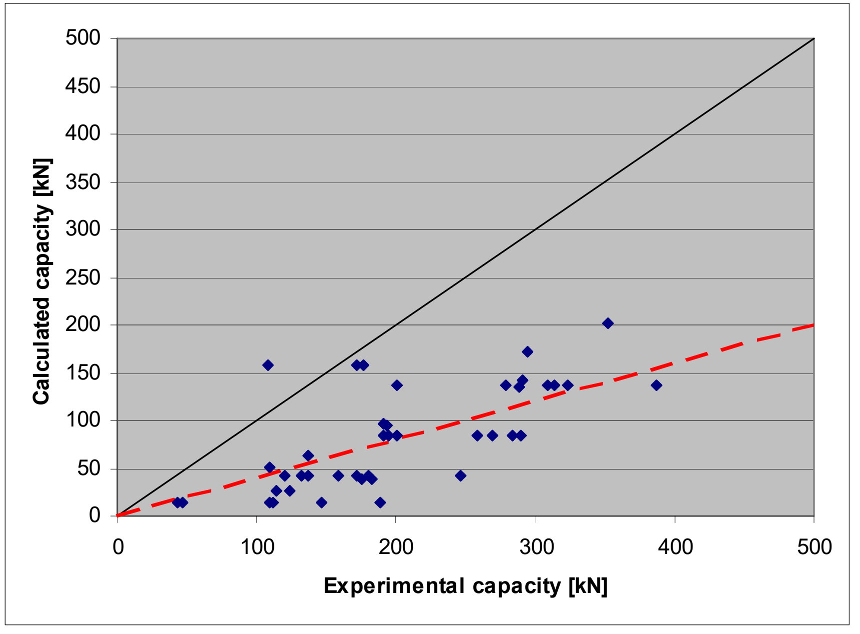
$$\mathrm{V}_Y = \sqrt{V_\Delta^2 + V_\varepsilon^2 + V_t^2}$$

Model uncertainty - examples



small bias: 1.06

small COV: V_{20} =12%



large bias: 2.5

large COV: V_{20} =25%

Exercises, self-study and reading

Read / self-study:

- Faber pp 85-104
- JCSS PMC Model uncertainty
- EN 1990, Annex D

Exercise:

- Model uncertainty